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Extended Bressoud–Wei and Koike skew Schur function identities

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ABSTRACT

The Jacobi–Trudi identity expresses a skew Schur function as a determinant of complete symmetric functions. Bressoud and Wei extend this idea, introducing an integer parameter $t \geqslant -1$ and showing that signed sums of skew Schur functions of a certain shape are expressible once again as a determinant of complete symmetric functions. Koike provides a Jacobi–Trudi-style definition of universal rational characters of the general linear group and gives their expansion as a signed sum of products of Schur functions in two distinct sets of variables. Here we extend Bressoud and Wei's formula by including an additional parameter and extending the result to the case of all integer t. Then we introduce this parameter idea to the Koike formula, extending it in the same way. We prove our results algebraically using Laplace determinantal expansions.

1. Introduction

The Jacobi–Trudi identity is a well-known and useful symmetric function identity that expresses a Schur function as a determinant of complete symmetric functions. A natural extension to this is a set of identities of Weyl [11] each of which expresses certain signed sum of skew Schur functions as a determinant of sums or differences of complete symmetric functions. Bressoud and Wei [1] extend the result and prove it combinatorially. In a similar vein, Koike [6] proves an identity for determinants of complete symmetric functions and sums of Schur functions involving two sets of variables. Here we generalize both of these approaches.

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Let m be a fixed positive integer and let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ be a sequence of m indeterminates. Let λ and σ be partitions of lengths $\ell(\lambda), \ell(\sigma) \leqslant m$ such that $\sigma \subseteq \lambda$. Then it is well known that the skew Schur function $s_{\lambda/\sigma}(\mathbf{x})$ satisfies the Jacobi–Trudi identity

$$s_{\lambda/\sigma}(\mathbf{x}) = \left|h_{\lambda_i-\sigma_j-i+j}(\mathbf{x})\right|, \tag{1}$$

where $h_k(\mathbf{x}) = 0$ for $k < 0$, and $h_k(\mathbf{x})$ is the complete homogeneous symmetric function of total degree k in the components of $\mathbf{x}$ for $k \geqslant 0$. The determinant on the right of (1) is an $m \times m$ determinant, so that $1 \leqslant i, j \leqslant m$. Its (i, j)th element, that is to say the element in the ith row and jth column, has been displayed. It is to be noted that $s_{\lambda/\sigma} = 0$ if $\sigma \nsubseteq \lambda$.

In 1992, Bressoud and Wei [1] proved that for all partitions λ of length $\ell(\lambda) \leqslant m$ and all integers $t \geqslant -1$ one has

$$\begin{aligned} &2^{(t-|t|)/2}\left|h_{\lambda_i-i+j}(\mathbf{x}) + (-1)^{(t+|t|)/2} h_{\lambda_i-i-j+1-t}(\mathbf{x})\right| \\ &\quad = \sum_{\sigma\in\mathcal{P}_t} (-1)^{[|\sigma|+r(\sigma)(|t|-1)]/2} s_{\lambda/\sigma}(\mathbf{x}), \end{aligned} \tag{2}$$

where the determinant on the left is again an $m \times m$ determinant, and on the right $|\sigma|$ is the weight of σ and the summation is over all partitions σ in the set $\mathcal{P}_t$ of rank $r(\sigma) \leqslant m$. This is the set of partitions, that when written in Frobenius notation, take the form

$$\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \quad \text{with } a_k - b_k = t \text{ for all } k = 1, 2, \ldots, r. \tag{3}$$

Here our first main result is to show that this result may be extended:

First Result. For all partitions λ of length $\ell(\lambda) \leqslant m$, for all integers t and any indeterminate q we have

$$\left|h_{\lambda_i-i+j}(\mathbf{x}) + q\chi_{j>-t} h_{\lambda_i-i-j+1-t}(\mathbf{x})\right| = \sum_{\sigma\in\mathcal{P}_t} (-1)^{[|\sigma|-r(\sigma)(t+1)]/2} q^{r(\sigma)} s_{\lambda/\sigma}(\mathbf{x}), \tag{4}$$

where the determinant on the left is an $m \times m$ determinant, χ_P is the truth function [2] defined to be 1 if the proposition P is true, and 0 otherwise, and the sum is over all partitions σ in the set $\mathcal{P}_t$ with $r(\sigma) \leqslant m + \chi_{t<0} t$.

The Bressoud–Wei identities of (2) can be recovered from (4) by setting $q = (-1)^t$ for all $t \geqslant 0$ and $q = 1$ for $t = -1$, with the factor $2^{(t-|t|)/2} = 2^{-1}$ in the case $t = -1$ compensating for the doubling of the entries in the first column of the Bressoud–Wei determinant as compared with those in the determinant of (4).

If we consider more than one set of variables, we can identify results with a similar flavour but with two sets of variables. In particular, for fixed positive integer m and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ a special case of the Jacobi–Trudi identity [7,8] is often used to define Schur functions $s_\lambda(\mathbf{x})$, for partitions λ with $\ell(\lambda) \leqslant m$, as follows:

$$s_\lambda(\mathbf{x}) = \left|h_{\lambda_i-i+j}(\mathbf{x})\right|. \tag{5}$$

It is less well known, that for fixed positive integers m and n, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, certain universal rational characters $s_{\lambda;\overline{\mu}}(\mathbf{x}, \mathbf{y})$, introduced by Koike [6, Definition 2.1], may be expressed, for pairs of partitions λ and μ with $\ell(\lambda) \leqslant m$ and $\ell(\mu) \leqslant n$, in terms of complete homogeneous symmetric functions by means of the following formula [6, Proposition 2.8]:

$$s_{\lambda;\overline{\mu}}(\mathbf{x}, \mathbf{y}) = \begin{vmatrix} h_{\mu_{n+1-i}+i-j}(\mathbf{y}) \\ \cdots \\ h_{\lambda_{i-n}-i+j}(\mathbf{x}) \end{vmatrix}, \tag{6}$$

where the determinant on the right is now an $(n+m)\times(n+m)$ determinant, so that $1\leqslant i, j\leqslant n+m$. It is partitioned immediately after its nth row. The expansion of this in terms of Schur functions in $\mathbf{x}$ and $\mathbf{y}$, takes the form

$$\begin{vmatrix} h_{\mu_{n+1-i}+i-j}(\mathbf{y}) \\ \cdots \\ h_{\lambda_{i-n}-i+j}(\mathbf{x}) \end{vmatrix} = \sum_{\zeta}(-1)^{|\zeta|} s_{\lambda/\zeta}(\mathbf{x}) s_{\mu/\zeta'}(\mathbf{y}), \tag{7}$$

since [6, Theorem 2.3]

$$s_{\lambda;\overline{\mu}}(\mathbf{x},\mathbf{y}) = \sum_{\zeta}(-1)^{|\zeta|} s_{\lambda/\zeta}(\mathbf{x}) s_{\mu/\zeta'}(\mathbf{y}). \tag{8}$$

In the case $m=n$ and $\mathbf{y}=\bar{\mathbf{x}}$ with $y_k=\bar{x}_k=x_k^{-1}$ for $k=1,2,\ldots,m$, these results correspond to the identity [6,5]:

$$s_{\lambda;\overline{\mu}}(\mathbf{x},\bar{\mathbf{x}}) = \sum_{\zeta}(-1)^{|\zeta|} s_{\lambda/\zeta}(\mathbf{x}) s_{\mu/\zeta'}(\bar{\mathbf{x}}) = \begin{vmatrix} h_{\mu_{n+1-i}+i-j}(\bar{\mathbf{x}}) \\ \cdots \\ h_{\lambda_{i-n}-i+j}(\mathbf{x}) \end{vmatrix}, \tag{9}$$

which has been used to define the rational [10], or mixed tensor [4] characters $s_{\lambda;\overline{\mu}}(\mathbf{x})$ of the general linear group $GL(m)$ of highest weight

$$(\lambda;\overline{\mu}) = (\lambda_1,\lambda_2,\ldots,\lambda_{\ell(\lambda)},0,\ldots,0,-\mu_{\ell(\mu)},\ldots,-\mu_2,-\mu_1). \tag{10}$$

Here, our second main result is to extend Koike's identity (7) in the same way that Bressoud and Wei extended the Jacobi–Trudi identity.

Second Result. First, let m and n be fixed positive integers, and let $\mathbf{x}=(x_1,\ldots,x_m)$ and $\mathbf{y}=(y_1,\ldots,y_n)$. Then for all partitions λ and μ of lengths $\ell(\lambda)\leqslant m$ and $\ell(\mu)\leqslant n$, for all integers p and q, and all indeterminates u and v, we have

$$\begin{vmatrix} h_{\mu_{n+1-i}+i-j}(\mathbf{y}) & \vdots & \chi_{j>n-q}\, u h_{\mu_{n+1-i}+i-j-q}(\mathbf{y}) \\ \cdots & & \cdots \\ \chi_{j\leqslant n+p}\, v h_{\lambda_{i-n}-i+j-p}(\mathbf{x}) & \vdots & h_{\lambda_{i-n}-i+j}(\mathbf{x}) \end{vmatrix}$$
$$= \sum_{\zeta\subseteq n^m}(-1)^{|\zeta|}(uv)^r s_{\lambda/(\zeta+p^r)}(\mathbf{x}) s_{\mu/(\zeta'+q^r)}(\mathbf{y}) \tag{11}$$

where $r=r(\zeta)$ and the $(n+m)\times(n+m)$ determinant on the left is partitioned immediately after the nth row and nth column, and $\sigma+\tau$, for any pair of partitions σ and τ, signifies the partition whose kth part is $\sigma_k+\tau_k$ for all k [8, p. 5].

For the two results (4) and (11), we give an algebraic proof involving the Laplace expansion of partitioned determinants and the link with a set of Schur functions specified by partitions which are rather simply expressed in Frobenius notation.

The structure of the paper is as follows. Section 2 covers background and notation. Section 3 extends the Bressoud and Wei formula to a larger range of parameters, and includes a proof of the new result. Section 4 provides a skew extension of the Koike identity and also includes its algebraic proof Section 5 involves some concluding remarks, particularly the use of the identities in providing determinantal expressions for classical group characters. Lattice path combinatorial proofs of our results are deferred to a future paper [3].

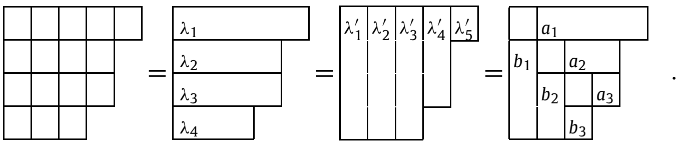
2. Notational and other preliminaries

2.1. Partitions, Frobenius notation and Young diagrams

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0$ be a partition of weight $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ and length $\ell(\lambda) = \max\{k \mid \lambda_k > 0\}$. Each such partition defines a Young or Ferrers diagram F^λ consisting of $|\lambda|$ boxes arranged in $\ell(\lambda)$ rows of lengths λ_i for $i = 1, 2, \ldots, \ell(\lambda)$. The conjugate partition λ' is the partition defined by the columns of F^λ of lengths λ'_j for $j = 1, 2, \ldots, \ell(\lambda')$ with $\ell(\lambda') = \lambda_1$. If F^λ has r boxes on the main diagonal, with arm and leg lengths $a_k = \lambda_k - k \geqslant 0$ and $b_k = \lambda'_k - k \geqslant 0$ for $k = 1, 2, \ldots, r$, then λ is written in Frobenius notation in the form

$$\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}, \tag{12}$$

with $a_1 > a_2 > \cdots > a_r \geqslant 0$ and $b_1 > b_2 > \cdots > b_r \geqslant 0$. Then λ is said to have Frobenius rank, or simply rank $r(\lambda) = r$. In the rank 3 case, this is illustrated by:



(13)

There exist a number of special families of partitions. Let $\mathcal{P}$ be the set of all partitions, including the zero partition $\lambda = 0 = (0, 0, \ldots, 0)$. This is the unique partition of weight, length and rank zero, that is $|0| = \ell(0) = r(0) = 0$. Then for any integer t let

$$\mathcal{P}_t = \left\{ \lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in \mathcal{P} \;\middle|\; a_k - b_k = t \;\; \begin{matrix} \text{for } k = 1, 2, \ldots, r \\ \text{and } r = 0, 1, \ldots \end{matrix} \right\}. \tag{14}$$

Here, as a matter of convention, it is to be assumed that the zero partition belongs to $\mathcal{P}_t$ for all integers t.

In what follows we shall also require the notion of skew Young diagrams. Given a pair of partitions λ and μ such that all boxes of F^μ are contained in F^λ we write $\mu \subseteq \lambda$. Removing the boxes of F^μ from F^λ leaves what is known as the skew Young diagram $F^{\lambda/\mu}$. For example, if $\lambda = (5, 4, 2)$ and $\mu = (3, 1)$, then

$$F^{\lambda/\mu} = \begin{ytableau} \none & \none & \none & \; & \; \\ \none & \; & \; & \; \\ \; & \; \end{ytableau}\,. \tag{15}$$

2.2. Schur and skew Schur functions

A semistandard Young tableau of shape λ or skew shape λ/σ is a filling of the boxes of F^λ or $F^{\lambda/\sigma}$ with non-negative integer entries, one in each box, such that the entries are weakly increasing from left to right across each row and strictly increasing from top to bottom down each column. Typically, for $\lambda = (5, 4, 2)$ and $\sigma = (3, 1)$ we have

$$\begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 4 \\ 3 & 4 \end{ytableau} \quad \text{and} \quad \begin{ytableau} \none & \none & \none & 2 & 4 \\ \none & 1 & 3 & 3 \\ 2 & 3 \end{ytableau}\,. \tag{16}$$

We denote the sets of all such semistandard Young tableaux T with entries no greater than m by $\mathcal{T}^\lambda(m)$ and $\mathcal{T}^{\lambda/\sigma}(m)$, as appropriate. The weight of any semistandard tableau T with entries no greater than m is defined in terms of indeterminates $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ by

$$\mathbf{x}(T) = x_1^{w_1(T)} x_2^{w_2(T)} \cdots x_m^{w_m(T)}, \tag{17}$$

where $w_k(T)$ is the number of entries k in T for $k = 1, 2, \ldots, m$. Our examples in (16) are of weights $x_1^3 x_2^2 x_3^3 x_4^3$ and $x_1 x_2^2 x_3^3 x_4$.

With this notation, the Schur function $s_\lambda(\mathbf{x})$ and the skew Schur function $s_{\lambda/\sigma}(\mathbf{x})$ are symmetric functions of the various x_i. They may be defined combinatorially by:

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathcal{T}^\lambda(m)} \mathbf{x}(T) \quad \text{and} \quad s_{\lambda/\sigma}(\mathbf{x}) = \sum_{T \in \mathcal{T}^{\lambda/\sigma}(m)} \mathbf{x}(T). \tag{18}$$

2.3. *Jacobi–Trudi identities*

The complete symmetric functions $h_k(\mathbf{x})$ are defined for all positive integers k by

$$h_k(\mathbf{x}) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant m} x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{19}$$

This is nothing other than the Schur function $s_k(\mathbf{x})$ specified by the one part partition $(k, 0, \ldots, 0)$, for which the corresponding semistandard tableaux, $T \in \mathcal{T}^{(k)}(m)$, take the typical one rowed form

$$\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{3}\,. \tag{20}$$

It is well known that there exists a multiplicative basis of the ring Λ_m of symmetric functions that is provided by the products $h_\mu(\mathbf{x}) = h_{\mu_1}(\mathbf{x}) h_{\mu_2}(\mathbf{x}) \cdots h_{\mu_m}(\mathbf{x})$, where μ ranges over the set of all partitions of lengths $\ell(\mu) \leqslant m$. The expansion of an arbitrary Schur function $s_\lambda(\mathbf{x})$ in terms of this basis is given by the Jacobi–Trudi identity:

$$s_\lambda(\mathbf{x}) = \left| h_{\lambda_i - i + j}(\mathbf{x}) \right|, \tag{21}$$

where $h_k(\mathbf{x}) = 0$ for $k < 0$ and $h_0(\mathbf{x}) = 1$. More generally, for partitions λ and σ such that $\sigma \subseteq \lambda$, the skew Schur function $s_{\lambda/\sigma}(\mathbf{x})$ can be expressed in the form

$$s_{\lambda/\sigma}(\mathbf{x}) = \left| h_{\lambda_i - \sigma_j - i + j}(\mathbf{x}) \right|. \tag{22}$$

2.4. *Modification rules*

The right-hand side of the Jacobi–Trudi identity (21) remains well defined even if the partition λ is replaced by the more general $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_m)$ with each κ_i an arbitrary integer for all $i = 1, 2, \ldots, m$. In this case, we can define

$$s_\kappa(\mathbf{x}) = \left| h_{\kappa_i - i + j}(x) \right|. \tag{23}$$

Here for given κ either $s_\kappa(\mathbf{x}) = 0$ or $s_\kappa(\mathbf{x}) = \pm s_\lambda(\mathbf{x})$ for some partition λ.

In fact, permuting rows in the determinant of (23) leads to a variety of identities. Such modification rules appear in Littlewood [7] and Murnaghan [9]. In particular, if $\kappa = (\kappa_1, \ldots, \kappa_j, \kappa_{j+1}, \ldots, \kappa_m)$ with $1 \leqslant j < m$, we have the identity

$$s_\kappa(\mathbf{x}) = -s_\mu(\mathbf{x}) \quad \text{where } \mu = (\kappa_1, \ldots, \kappa_{j+1} - 1, \kappa_j + 1, \ldots, \kappa_n). \tag{24}$$

Iterating this yields

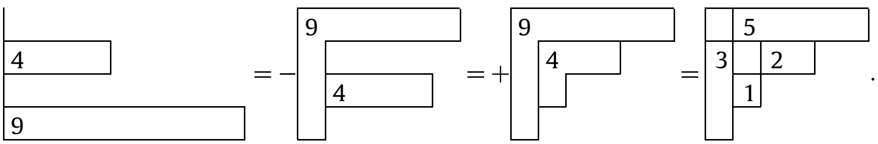
$$s_\kappa(\mathbf{x}) = (-1)^{j-1} s_\nu(\mathbf{x}) \quad \text{where } \nu = (\kappa_{j+1} - j, \kappa_1 + 1, \ldots, \kappa_j + 1, \kappa_{j+2}, \ldots, \kappa_n). \tag{25}$$

These identities are sometimes referred to as modification or standardization rules.

For example, if $m=4$ and $\kappa=(0,4,0,9)$ then $s_{0409}(\mathbf{x})=-s_{6151}(\mathbf{x})=s_{6421}(\mathbf{x})$, as can be seen directly from the determinantal identities

$$
\begin{aligned}
s_{0409}(\mathbf{x}) &= \begin{vmatrix} 1 & h_1(\mathbf{x}) & h_2(\mathbf{x}) & h_3(\mathbf{x}) \\ h_3(\mathbf{x}) & h_4(\mathbf{x}) & h_5(\mathbf{x}) & h_6(\mathbf{x}) \\ 0 & 0 & 1 & h_1(\mathbf{x}) \\ h_6(\mathbf{x}) & h_7(\mathbf{x}) & h_8(\mathbf{x}) & h_9(\mathbf{x}) \end{vmatrix} \\
&= \begin{vmatrix} h_6(\mathbf{x}) & h_7(\mathbf{x}) & h_8(\mathbf{x}) & h_9(\mathbf{x}) \\ 1 & h_1(\mathbf{x}) & h_2(\mathbf{x}) & h_3(\mathbf{x}) \\ h_3(\mathbf{x}) & h_4(\mathbf{x}) & h_5(\mathbf{x}) & h_6(\mathbf{x}) \\ 0 & 0 & 1 & h_1(\mathbf{x}) \end{vmatrix} = -s_{6151}(\mathbf{x}) \\
&= \begin{vmatrix} h_6(\mathbf{x}) & h_7(\mathbf{x}) & h_8(\mathbf{x}) & h_9(\mathbf{x}) \\ h_3(\mathbf{x}) & h_4(\mathbf{x}) & h_5(\mathbf{x}) & h_6(\mathbf{x}) \\ 1 & h_1(\mathbf{x}) & h_2(\mathbf{x}) & h_3(\mathbf{x}) \\ 0 & 0 & 1 & h_1(\mathbf{x}) \end{vmatrix} = +s_{6421}(\mathbf{x}).
\end{aligned} \tag{26}
$$

This same modification may also be accomplished diagrammatically. One takes the non-standard Young diagram F^κ with left-adjusted row lengths specified by the parts of κ, and standardizes it by converting the horizontal rows into nested hooks of lengths $\kappa_1, \kappa_2, \ldots, \kappa_m$ while displacing the remainder of the diagram diagonally. A sign change is introduced for each new row beyond the first that each continuous strip occupies. Of course, this does not always lead to a regular or standard Young diagram, but in our example it does do so, as illustrated by the diagrams:



(27)

In the final diagram the nested continuous strips of lengths 9 and 4 have been relabelled in terms of their arm and leg lengths so as to illustrate more clearly the fact that in Frobenius notation

$$
s_\kappa(\mathbf{x}) = s_{0409}(\mathbf{x}) = s_{6421}(\mathbf{x}) = s_\lambda(\mathbf{x}) \quad \text{with } \lambda = (6,4,2,1) = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}. \tag{28}
$$

More generally, if κ_j is nonzero for $j \in \{b_1+1, b_2+1, \ldots, b_r+1\}$, and zero otherwise, then without loss of generality we may choose $b_1 > b_2 > \cdots > b_r \geqslant 0$, and if $j = b_k+1$, we may set $\kappa_j = a_k + b_k + 1$ so that

$$
\kappa = \left(0^{b_r}, a_r + b_r + 1, 0^{b_{r-1}-b_r-1}, \ldots, a_2 + b_2 + 1, 0^{b_1-b_2-1}, a_1 + b_1 + 1\right). \tag{29}
$$

Then, provided that $a_1 > a_2 > \cdots > a_r \geqslant 0$, we have

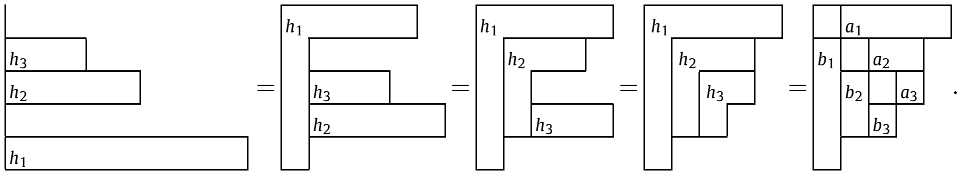
$$
s_\kappa(\mathbf{x}) = (-1)^{b_1+b_2+\cdots+b_r} s_\sigma(\mathbf{x}) \tag{30}
$$

with

$$
\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \tag{31}
$$

and $r = r(\sigma)$.

This can be seen schematically from the following diagrams corresponding to the case $\kappa = (0, h_3, h_2, 0, h_1)$, with $h_k = a_k + b_k + 1$ for $k = 1,2,3$, and $r = 3 = r(\sigma)$.



(32)

Modification rules may be applied in exactly the same way to non-standard skew Schur functions. In particular, for any partition λ of length $\ell(\lambda) \leqslant m$ and any $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_m)$ such that $s_\kappa(\mathbf{x}) = \pm s_\sigma(\mathbf{x})$ for some partition σ (necessarily having length $\ell(\sigma) \leqslant m$, by permuting columns this time rather than rows) one obtains

$$s_{\lambda/\kappa}(\mathbf{x}) = \left|h_{\lambda_i-\kappa_j-i+j}(\mathbf{x})\right| = \pm\left|h_{\lambda_i-\sigma_j-i+j}(\mathbf{x})\right| = \pm s_{\lambda/\sigma}(\mathbf{x}). \tag{33}$$

3. Extended Bressoud–Wei identities

As pointed out in the Introduction, Bressoud and Wei's original result (2) states that for all $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, partitions λ of length $\ell(\lambda) \leqslant m$ and all integers $t \geqslant -1$,

$$2^{(t-|t|)/2}\left|h_{\lambda_i-i+j}(\mathbf{x}) + (-1)^{(t+|t|)/2} h_{\lambda_i-i-j+1-t}(\mathbf{x})\right| = \sum_{\sigma\in\mathcal{P}_t} (-1)^{[|\sigma|+r(\sigma)(|t|-1)]/2} s_{\lambda/\sigma}(\mathbf{x}), \tag{34}$$

where the determinant on the left is $m \times m$, and on the right the summation is over all partitions σ of weight $|\sigma|$ and rank $r = r(\sigma)$ in the set $\mathcal{P}_t$. They note that for $t = -1, 0, 1$ this simplifies to yield results first due to Weyl [11]:

$$\begin{aligned}
&\frac{1}{2}\left|h_{\lambda_i-i+j}(\mathbf{x}) + h_{\lambda_i-i-j+2}(\mathbf{x})\right| = \sum_{\sigma\in\mathcal{P}_{-1}} (-1)^{|\sigma|/2} s_{\lambda/\sigma}(\mathbf{x});\\
&\left|h_{\lambda_i-i+j}(\mathbf{x}) + h_{\lambda_i-i-j+1}(\mathbf{x})\right| = \sum_{\sigma\in\mathcal{P}_0} (-1)^{(|\sigma|-r)/2} s_{\lambda/\sigma}(\mathbf{x});\\
&\left|h_{\lambda_i-i+j}(\mathbf{x}) - h_{\lambda_i-i-j}(\mathbf{x})\right| = \sum_{\sigma\in P_1} (-1)^{|\sigma|/2} s_{\lambda/\sigma}(\mathbf{x}).
\end{aligned} \tag{35}$$

The proof of Bressoud and Wei uses lattice path techniques involving two sets of possible starting points, Ps and P's. We extend their technique to cover cases for which these two sets would, without any additional constraint, intersect at more than one point. In doing so we extend their results to a larger selection of values of t, in fact to all integer values. We offer an algebraic proof, with a combinatorial proof to be made available elsewhere [3].

The main result to be established here is the following:

Theorem 1. *Let m be a fixed positive integer, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ a sequence of indeterminates, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ a partition of length $\ell(\lambda) \leqslant m$. Then for all integers t and all q we have*

$$\left|h_{\lambda_i-i+j}(\mathbf{x}) + q\chi_{j>-t} h_{\lambda_i-i-j+1-t}(\mathbf{x})\right| = \sum_{\sigma\in\mathcal{P}_t} (-1)^{[|\sigma|-r(\sigma)(t+1)]/2} q^{r(\sigma)} s_{\lambda/\sigma}(\mathbf{x}), \tag{36}$$

where the determinant is $m \times m$.

Proof. Expanding the determinant gives

$$\left|h_{\lambda_i-i+j}(\mathbf{x}) + q\chi_{j>-t} h_{\lambda_i-i-j+1-t}(\mathbf{x})\right| = \sum_{r=0}^{m} \sum_{\kappa} q^r \left|h_{\lambda_i-i+j-\kappa_j}(\mathbf{x})\right|. \tag{37}$$

Here r indicates the number of columns of the original determinant from which elements involving the factor q have been retained. For each such r, the sum is over those κ such that $\kappa_j = 2j - 1 + t$ for

$j \in \{j_1, j_2, \ldots, j_r\}$ and $\kappa_j = 0$ otherwise. Without loss of generality, $m \geqslant j_1 > j_2 > \cdots > j_r \geqslant 1 - \chi_{t<0} t$. Since $\kappa = (0, \ldots, 0, 2j_r - 1 + t, 0, \ldots, 0, 2j_2 - 1 + t, 0, \ldots, 0, 2j_1 - 1 + t, 0, \ldots, 0)$ we may permute columns of $|h_{\lambda_i - i + j - \kappa_j}(\mathbf{x})|$ to give $(-1)^{j_1 - 1 + j_2 - 1 + \cdots + j_r - 1} |h_{\lambda_i - i + j - \sigma_j}(\mathbf{x})|$, with

$$\sigma = \begin{pmatrix} j_1 - 1 + t & j_2 - 1 + t & \cdots & j_r - 1 + t \\ j_1 - 1 & j_2 - 1 & \cdots & j_r - 1 \end{pmatrix} \in \mathcal{P}_t \tag{38}$$

and $r = r(\sigma)$. The fact that $|h_{\lambda_i - i + j - \sigma_j}(\mathbf{x})| = s_{\lambda/\sigma}(\mathbf{x})$ and $|\sigma| = 2(j_1 - 1 + j_2 - 1 + \cdots + j_r - 1) + r(t+1)$ then completes the proof of (36). □

4. Skew extension of the Koike identity

In his seminal paper [6], Koike derives a number of decomposition formulas for tensor products of various representations of the classical groups. In this paper he derives also a number of determinantal results, including the one we extend here. His proof techniques are exclusively algebraic. His result of particular interest here involves two sets of variables – or two alphabets – and expresses a signed sum of a product of skew Schur functions as a determinant of complete symmetric functions, i.e.

$$\begin{vmatrix} h_{\mu_{n+1-i}+i-j}(\mathbf{y}) \\ \cdots \\ h_{\lambda_{i-n}-i+j}(\mathbf{x}) \end{vmatrix} = \sum_{\zeta} (-1)^{|\zeta|} s_{\lambda/\zeta}(\mathbf{x}) s_{\mu/\zeta'}(\mathbf{y}). \tag{39}$$

This can be viewed as an analogue for two sets of variables of the Jacobi–Trudi identity (21) involving a single set of variables. It is natural therefore to seek some sort of analogue of our generalized Bressoud and Wei identity of Theorem 1 by using the techniques of the previous section in a two alphabet context. Such a result takes the form of the following generalization of Koike's identity:

Theorem 2. *For fixed positive integers m and n, let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two sequences of indeterminates, and let λ and μ be a pair of partitions of lengths $\ell(\lambda) \leqslant m$ and $\ell(\mu) \leqslant n$. Then for each pair of integers p and q, and all u and v, we have*

$$\left|\begin{array}{c:c} h_{\mu_{n+1-i}+i-j}(\mathbf{y}) & \chi_{j>n-q}\, u h_{\mu_{n+1-i}+i-j-q}(\mathbf{y}) \\ \cdots & \cdots \\ \chi_{j\leqslant n+p}\, v h_{\lambda_{i-n}-i+j-p}(\mathbf{x}) & h_{\lambda_{i-n}-i+j}(\mathbf{x}) \end{array}\right|$$

$$= \sum_{\zeta \subseteq n^m} (-1)^{|\zeta|} (uv)^r s_{\lambda/(\zeta+p^r)}(\mathbf{x}) s_{\mu/(\zeta'+q^r)}(\mathbf{y}) \tag{40}$$

where $r = r(\zeta)$ and the $(n+m) \times (n+m)$ determinant is partitioned immediately after the nth row and nth column. If $\zeta \subseteq (n^m)$ is given in Frobenius notation by $\zeta = \left(\begin{smallmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{smallmatrix}\right)$, with $n > a_1 > a_2 > \cdots > a_r$ and $m > b_1 > b_2 > \cdots > b_r$, then:

$$\zeta + p^r = \begin{pmatrix} a_1 + p & a_2 + p & \cdots & a_r + p \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}; \tag{41}$$

and

$$\zeta' + q^r = \begin{pmatrix} b_1 + q & b_2 + q & \cdots & b_r + q \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}, \tag{42}$$

with $a_r \geqslant \max\{0, -p\}$ and $b_r \geqslant \max\{0, -q\}$.

Before embarking on the proof it might be noted that the structure of the partitioned matrix is exemplified in the case $m=3$, $n=4$, $p=-2$, $q=-1$, $\lambda=(5,3,2)$ and $\mu=(4,3,2,2)$ by

$$\left|\begin{array}{cccc|ccc} h_2(\mathbf{y}) & h_1(\mathbf{y}) & 1 & 0 & 0 & 0 & 0 \\ h_3(\mathbf{y}) & h_2(\mathbf{y}) & h_1(\mathbf{y}) & 1 & 0 & 0 & 0 \\ h_5(\mathbf{y}) & h_4(\mathbf{y}) & h_3(\mathbf{y}) & h_2(\mathbf{y}) & 0 & uh_1(\mathbf{y}) & u \\ h_7(\mathbf{y}) & h_6(\mathbf{y}) & h_5(\mathbf{y}) & h_4(\mathbf{y}) & 0 & uh_3(\mathbf{y}) & uh_2(\mathbf{y}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ vh_3(\mathbf{x}) & vh_4(\mathbf{x}) & 0 & 0 & h_5(\mathbf{x}) & h_6(\mathbf{x}) & h_7(\mathbf{x}) \\ v & vh_1(\mathbf{x}) & 0 & 0 & h_2(\mathbf{x}) & h_3(\mathbf{x}) & h_4(\mathbf{x}) \\ 0 & 0 & 0 & 0 & 1 & h_1(\mathbf{x}) & h_2(\mathbf{x}) \end{array}\right|. \tag{43}$$

Proof. The Laplace expansion of the determinant leads to

$$\begin{aligned} &\left|\begin{array}{c|c} h_{\mu_{n+1-i}+i-j}(\mathbf{y}) & \chi_{j>n-q}\, u h_{\mu_{n+1-i}+i-j-q}(\mathbf{y}) \\ \cdots & \cdots \\ \chi_{j\leqslant n+p}\, v h_{\lambda_{i-n}-i+j-p}(\mathbf{x}) & h_{\lambda_{i-n}-i+j}(\mathbf{x}) \end{array}\right| \\ &\quad = \sum_{\pi\in S_{n+m}\backslash S_m\times S_n} (-1)^{\pi}(uv)^r \left|d_j h_{\mu_{n+1-i}+i-j-q_j}(\mathbf{y})\right|_{i\in N,\, j\in N_\pi} \left|c_j h_{\lambda_{i-n}-i+j-p_j}(\mathbf{x})\right|_{i\in M^n,\, j\in M^n_\pi} \end{aligned} \tag{44}$$

where

$$\begin{aligned} N &= \{1,2,\ldots,n\}, & N_\pi &= \bigl\{\pi(j)\mid j\in N\bigr\}, \\ M^n &= \{n+1,n+2,\ldots,n+m\}, & M^n_\pi &= \bigl\{\pi(j)\mid j\in M^n\bigr\} \end{aligned} \tag{45}$$

and

$$r = \#\bigl\{j\in N_\pi \mid j\in M^n\bigr\} = \#\bigl\{j\in M^n_\pi \mid j\in N\bigr\}, \tag{46}$$

while

$$p_j = \begin{cases} p & \text{if } j\in N; \\ 0 & \text{if } j\in M^n, \end{cases} \quad\text{and}\quad q_j = \begin{cases} 0 & \text{if } j\in N; \\ q & \text{if } j\in M^n. \end{cases} \tag{47}$$

and

$$c_j = \begin{cases} 1 & \text{if } n+p\geqslant j\in N; \\ 0 & \text{if } n+p< j\in N; \\ 1 & \text{if } j\in M^n, \end{cases} \quad\text{and}\quad d_j = \begin{cases} 1 & \text{if } j\in N; \\ 0 & \text{if } n-q\geqslant j\in M^n; \\ 1 & \text{if } n-q< j\in M^n. \end{cases} \tag{48}$$

The coset representatives may be chosen in such a way that

$$\pi(1)<\pi(2)<\cdots<\pi(n) \quad\text{and}\quad \pi(n+1)<\pi(n+2)<\cdots<\pi(n+m). \tag{49}$$

Each such permutation π may be written in the form

$$\pi = \begin{pmatrix} 1 & \cdots & n-1 & n & n+1 & n+2 & \cdots & n+m \\ 1+\zeta'_n & \cdots & n-1+\zeta'_2 & n+\zeta'_1 & n+1-\zeta_1 & n+2-\zeta_2 & \cdots & n+m-\zeta_m \end{pmatrix} \tag{50}$$

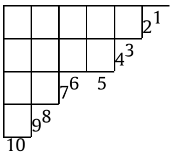
for some partition $\zeta\subseteq(n^m)$. Indeed, every such $\zeta\subseteq(n^m)$ arises in this way since there exists a bijective map from the coset representatives π to the partitions $\zeta\subseteq(n^m)$. This is constructed by labelling the consecutive boundary edges of $F^\zeta \subseteq F^{(n^m)}$ with integers $j = 1, 2, \dots, n+m$, with the edge labelled j either horizontal or vertical according as $\pi^{-1}(j)$ is either $\leqslant n$ or $> n$, respectively. Moreover, the rank $r(\zeta)$ of ζ is the maximum k such that $\pi(n+k) = n+k-\zeta_k \leqslant n$, or equivalently $\pi(n-k+1) = n-k+1+\zeta'_k > n$, and by counting descents

$$(-1)^\pi = (-1)^{\zeta'_n+\cdots+\zeta'_2+\zeta'_1} = (-1)^{|\zeta|}. \tag{51}$$

For example, if $m=4$, $n=6$ and

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 3 & 5 & 6 & 8 & 10 & 2 & 4 & 7 & 9 \end{pmatrix}$$

then from (50) it follows that $\zeta = (5,4,2,1)$ and $\zeta' = (4,3,2,2,1,0)$, with $r(\zeta) = 2$ and $(-1)^\pi = (-1)^{1+2+2+3+4} = +1$, while the corresponding map from ζ back to π is provided by the edge labelling:



(52)

Replacing the sum over $\pi \in S_{n+m} \setminus S_m \times S_n$ in (44) by a sum over $\zeta \subseteq (n^m)$ we obtain

$$\begin{aligned} &\sum_{\pi \in S_{n+m}\setminus S_m\times S_n} (-1)^\pi (uv)^r \left|d_j h_{\mu_{n+1-i}+i-j-q_j}(\mathbf{y})\right|_{i\in N,\, j\in N_\pi} \left|c_j h_{\lambda_{i-n}-i+j-p_j}(\mathbf{x})\right|_{i\in M^n,\, j\in M^n_\pi} \\ &\quad = \sum_{\zeta\subseteq(n^m)} (-1)^{|\zeta|} (uv)^{r(\zeta)} \left|h_{\mu_k-k+l-\tau_l}(\mathbf{y})\right|_{k,l\in N} \left|h_{\lambda_f-f+g-\sigma_g}(\mathbf{x})\right|_{f,g\in M}, \end{aligned} \tag{53}$$

where it has been convenient in the light of (50) to set $i = n-k+1$ and $j = n-l+1-\zeta'_l$ in the first determinant, and $i = n+f$ and $j = n+g-\zeta_g$ in the second determinant. From the prescribed values of c_j, d_j, p_j and q_j it then follows that the only non-zero terms are those for which

$$\sigma_g = \begin{cases} \zeta_g + p & \text{if } 1 \leqslant g \leqslant r; \\ \zeta_g & \text{if } r < g \leqslant m, \end{cases} \quad \text{and} \quad \tau_l = \begin{cases} \zeta' + q & \text{if } 1 \leqslant l \leqslant r; \\ \zeta' & \text{if } r < l \leqslant n. \end{cases} \tag{54}$$

with

$$\zeta_g - g + p \geqslant 0 \quad \text{and} \quad \zeta'_l - l + q \geqslant 0 \quad \text{for } 1 \leqslant g, l \leqslant r \tag{55}$$

by virtue of the constraints $j = n+g-\zeta_g \leqslant n+p$ and $j = n-l+1+\zeta'_l > n-q$ that apply for $1 \leqslant g, l \leqslant r$.

The definitions (54), together with the constraints (55), are just what are required to see that $\sigma = (\zeta + p^r)$ and $\tau = (\zeta' + q^r)$, as in (41) and (42), respectively, together with the constraints $a_r \geqslant \max\{0, -p\}$ and $b_r \geqslant \max\{0, -q\}$. The fact that $|h_{\mu_k-k+l-\tau_l}(\mathbf{y})| = s_{\mu/\tau}(\mathbf{y})$ and $|h_{\lambda_f-f+g-\sigma_g}(\mathbf{x})| = s_{\lambda/\sigma}(\mathbf{x})$ then serves to complete the proof of Theorem 2. $\square$

5. Closing remarks

5.1. Dual identities

The elementary symmetric functions $e_k(\mathbf{x})$ are defined for all positive integers k by

$$e_k(\mathbf{x}) = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant m} x_{i_1}x_{i_2}\cdots x_{i_k}. \tag{56}$$

An alternative basis of the ring Λ_m of symmetric functions is provided by the products $e_\mu(\mathbf{x}) = e_{\mu_1}(\mathbf{x})e_{\mu_2}(\mathbf{x})\cdots e_{\mu_m}(\mathbf{x})$, where μ ranges over all partitions of conjugate length $\ell(\mu') \leqslant m$. In terms of this basis we have the dual Jacobi–Trudi, or Naegelsbach–Kostka identity [7,8]

$$s_{\lambda/\sigma}(\mathbf{x}) = \left|e_{\lambda'_i-\sigma'_j-i+j}(\mathbf{x})\right|, \tag{57}$$

where $e_k(\mathbf{x}) = 0$ for $k < 0$ and $e_0(\mathbf{x}) = 1$.

This may be derived most easily by exploiting the involution ω on the ring Λ_m of symmetric functions [8] whose action is such that $\omega(h_k(\mathbf{x})) = e_k(\mathbf{x})$ and $\omega(s_\lambda(\mathbf{x})) = s_{\lambda'}(\mathbf{x})$, while more generally $\omega(s_{\lambda/\sigma}(\mathbf{x})) = s_{\lambda'/\sigma'}(\mathbf{x})$. Applying this to the Jacobi–Trudi identity (1) and replacing λ and σ by λ' and σ', respectively, immediately gives (57). Applying this same transformation to (4) one equally easily obtains the identity

$$\left|e_{\lambda'_i-i+j}(\mathbf{x}) + q\chi_{j>-t}e_{\lambda'_i-i-j+1-t}(\mathbf{x})\right| = \sum_{\sigma\in\mathcal{P}_{-t}} (-1)^{[|\sigma|-r(\sigma)(t+1)]/2}q^{r(\sigma)}s_{\lambda/\sigma}(\mathbf{x}), \tag{58}$$

where use has been made of the fact that conjugating partitions simply interchanges their arm and leg lengths which implies that if $\sigma \in \mathcal{P}_t$ then $\sigma' \in \mathcal{P}_{-t}$, and *vice versa*, for all integers t.

In precisely the same way, in the case of (7), the independent application of ω in each of the two rings Λ_m and Λ_n of symmetric functions in the components of $\mathbf{x}$ and of $\mathbf{y}$, respectively, yields the identity [6]

$$\left|\begin{matrix} e_{\mu'_{n+1-i}+i-j}(\mathbf{y}) \\ \cdots \\ e_{\lambda'_{i-n}-i+j}(\mathbf{x}) \end{matrix}\right| = \sum_\zeta (-1)^{|\zeta|}s_{\lambda/\zeta}(\mathbf{x})s_{\mu/\zeta'}(\mathbf{y}). \tag{59}$$

In fact it is the left-hand side of this that Koike uses to define $s_{\lambda;\overline{\mu}}(\mathbf{x},\mathbf{y})$. Proceeding in the same way in the case of (11) one obtains

$$\begin{aligned} &\left|\begin{matrix} e_{\mu'_{n+1-i}+i-j}(\mathbf{y}) & \vdots & \chi_{j>n-q}u e_{\mu'_{n+1-i}+i-j-q}(\mathbf{y}) \\ \cdots & & \cdots \\ \chi_{j\leqslant n+p}v e_{\lambda'_{i-n}-i+j-p}(\mathbf{x}) & \vdots & e_{\lambda'_{i-n}-i+j}(\mathbf{x}) \end{matrix}\right| \\ &\quad = \sum_{\zeta\subseteq n^m} (-1)^{|\zeta|}(uv)^r s_{\lambda/(\zeta+p^r)'}(\mathbf{x})s_{\mu/(\zeta+q^r)'}(\mathbf{y}). \end{aligned} \tag{60}$$

Although we have not checked the details, there is little doubt that combinatorial lattice path proofs of (58) and (60) can be readily established.

5.2. *Classical group characters*

It is well known [9,7,11,8] that for $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ the Schur function $s_\lambda(\mathbf{x})$ with $\ell(\lambda) \leqslant m$ is nothing other than the character of the irreducible representation $V^\lambda_{GL(m)}$ of $GL(m)$ having highest weight $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_m\epsilon_m$. That is to say for any group element $g \in GL(m)$ having eigenvalues $x_1, x_2, \ldots, x_m$

$$\mathrm{ch}\, V^\lambda_{GL(m)}(\mathbf{x}) = s_\lambda(\mathbf{x}) = \left|h_{\lambda_i-i+j}(\mathbf{x})\right| = \left|e_{\lambda'_i-i+j}(\mathbf{x})\right|. \tag{61}$$

We are in a position to gather together a collection of character formulas for the orthogonal and symplectic groups. With the notation $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, $\bar{\mathbf{x}} = (x_1^{-1}, x_2^{-1}, \ldots, x_m^{-1})$,

$$\Delta(\mathbf{x}) = \prod_{i=1}^m \left(x_i^{\frac12} + x_i^{-\frac12}\right) \quad \text{and} \quad \Delta'(\mathbf{x}) = \prod_{i=1}^m \left(x_i^{\frac12} - x_i^{-\frac12}\right), \tag{62}$$

these take the form:

$$\begin{aligned}
\mathrm{ch}\, V^{\lambda}_{SO(2m+1)}(\mathbf{x},\bar{\mathbf{x}},1) &= \sum_{\sigma\in\mathcal{P}_0} (-1)^{(|\sigma|-r(\sigma))/2} s_{\lambda/\sigma}(\mathbf{x},\bar{\mathbf{x}}) \qquad t=0, q=1 \\
&= \left|h_{\lambda_i-i+j}(\mathbf{x},\bar{\mathbf{x}}) + h_{\lambda_i-i-j+1}(\mathbf{x},\bar{\mathbf{x}})\right|; \qquad (63)\\
\mathrm{ch}\, V^{\lambda}_{O(2m+1)}(\mathbf{x},\bar{\mathbf{x}},-1) &= \sum_{\sigma\in\mathcal{P}_0} (-1)^{(|\sigma|+r(\sigma))/2} s_{\lambda/\sigma}(\mathbf{x},\bar{\mathbf{x}}) \qquad t=0, q=-1 \\
&= \left|h_{\lambda_i-i+j}(\mathbf{x},\bar{\mathbf{x}}) - h_{\lambda_i-i-j+1}(\mathbf{x},\bar{\mathbf{x}})\right|; \qquad (64)\\
\mathrm{ch}\, V^{(\lambda+\frac{1}{2}^m)}_{SO(2m+1)}(\mathbf{x},\bar{\mathbf{x}},1) &= \Delta(\mathbf{x}) \sum_{\sigma\in\mathcal{P}_{-1}} (-1)^{(|\sigma|/2} s_{\lambda/\sigma}(\mathbf{x},\bar{\mathbf{x}}) \qquad t=-1, q=1 \\
&= \Delta(\mathbf{x})\left|h_{\lambda_i-i+j}(\mathbf{x},\bar{\mathbf{x}}) + \chi_{j>1} h_{\lambda_i-i-j+2}(\mathbf{x},\bar{\mathbf{x}})\right|; \qquad (65)\\
\mathrm{ch}\, V^{\lambda}_{Sp(2m)}(\mathbf{x},\bar{\mathbf{x}}) &= \sum_{\sigma\in\mathcal{P}_{-1}} (-1)^{|\sigma|/2} s_{\lambda/\sigma}(\mathbf{x},\bar{\mathbf{x}}) \qquad t=-1, q=1 \\
&= \left|h_{\lambda_i-i+j}(\mathbf{x},\bar{\mathbf{x}}) + \chi_{j>1} h_{\lambda_i-i-j+2}(\mathbf{x},\bar{\mathbf{x}})\right|; \qquad (66)\\
\mathrm{ch}\, V^{\lambda}_{O(2m)}(\mathbf{x},\bar{\mathbf{x}}) &= \sum_{\sigma\in\mathcal{P}_{1}} (-1)^{|\sigma|/2} s_{\lambda/\sigma}(\mathbf{x},\bar{\mathbf{x}}) \qquad t=1, q=-1 \\
&= \left|h_{\lambda_i-i+j}(\mathbf{x},\bar{\mathbf{x}}) - h_{\lambda_i-i-j}(\mathbf{x},\bar{\mathbf{x}})\right|; \qquad (67)\\
\mathrm{ch}\, V^{\lambda}_{O(2m+2)}(\mathbf{x},\bar{\mathbf{x}},1,-1) &= \sum_{\sigma\in\mathcal{P}_{-1}} (-1)^{|\sigma|/2} s_{\lambda/\sigma}(\mathbf{x},\bar{\mathbf{x}}) \qquad t=-1, q=1 \\
&= \left|h_{\lambda_i-i+j}(\mathbf{x},\bar{\mathbf{x}}) + \chi_{j>1} h_{\lambda_i-i-j+2}(\mathbf{x},\bar{\mathbf{x}})\right|; \qquad (68)\\
\mathrm{ch}\, V^{(\lambda+1^m)'}_{SO(2m)}(\mathbf{x},\bar{\mathbf{x}}) &= \Delta(\mathbf{x})\Delta'(\mathbf{x}) \sum_{\sigma\in\mathcal{P}_{-1}} (-1)^{|\sigma|/2} s_{\lambda/\sigma}(\mathbf{x},\bar{\mathbf{x}}) \qquad t=-1, q=1 \\
&= \Delta(\mathbf{x})\Delta'(\mathbf{x})\left|h_{\lambda_i-i+j}(\mathbf{x},\bar{\mathbf{x}}) + \chi_{j>1} h_{\lambda_i-i-j+2}(\mathbf{x},\bar{\mathbf{x}})\right|; \qquad (69)\\
\mathrm{ch}\, V^{(\lambda+\frac{1}{2}^m)}_{SO(2m)}(\mathbf{x},\bar{\mathbf{x}}) &= \Delta(\mathbf{x}) \sum_{\sigma\in\mathcal{P}_0} (-1)^{(|\sigma|+r(\sigma))/2} s_{\lambda/\sigma}(\mathbf{x},\bar{\mathbf{x}}) \qquad t=0, q=-1 \\
&= \Delta'(\mathbf{x})\left|h_{\lambda_i-i+j}(\mathbf{x},\bar{\mathbf{x}}) - h_{\lambda_i-i-j+1}(\mathbf{x},\bar{\mathbf{x}})\right|; \qquad (70)\\
\mathrm{ch}\, V^{(\lambda+\frac{1}{2}^m)'}_{SO(2m)}(\mathbf{x},\bar{\mathbf{x}}) &= \Delta'(\mathbf{x}) \sum_{\sigma\in\mathcal{P}_0} (-1)^{(|\sigma|-r(\sigma))/2} s_{\lambda/\sigma}(\mathbf{x},\bar{\mathbf{x}}) \qquad t=0, q=1 \\
&= \Delta'(\mathbf{x})\left|h_{\lambda_i-i+j}(\mathbf{x},\bar{\mathbf{x}}) + h_{\lambda_i-i-j+1}(\mathbf{x},\bar{\mathbf{x}})\right|, \qquad (71)
\end{aligned}$$

where in each case λ is a partition of length $\ell(\lambda) \leqslant m$, and in the case of $SO(2m)$ primes $'$ are used to signify difference characters [7]. The corresponding irreducible characters are given by

$$\mathrm{ch}\, V^{(\lambda+\frac{1}{2}^m)\pm}_{SO(2m)}(\mathbf{x},\bar{\mathbf{x}}) = \frac{1}{2}\left(\mathrm{ch}\, V^{(\lambda+\frac{1}{2}^m)}_{SO(2m)}(\mathbf{x},\bar{\mathbf{x}}) \pm \mathrm{ch}\, V^{(\lambda+\frac{1}{2}^m)'}_{SO(2m)}(\mathbf{x},\bar{\mathbf{x}})\right). \qquad (72)$$

In each case the determinantal results follow from Theorem 1 for various, but not all, combinations of $t \in \{-1, 0, 1\}$ and $q = \pm 1$. It is a straightforward task, but rather tedious, to use the automorphism ω to express all the above characters as determinants in elementary rather than complete symmetric functions.

In the case of our two alphabet identities, the only one that we are aware of that corresponds directly to a character formula is the following [6,5]:

$$\begin{aligned} \operatorname{ch} V_{GL(m)}^{\lambda;\overline{\mu}}(\mathbf{x}) = s_{\lambda;\overline{\mu}}(\mathbf{x},\bar{\mathbf{x}}) &= \sum_{\zeta}(-1)^{|\zeta|} s_{\lambda/\zeta}(\mathbf{x}) s_{\mu/\zeta'}(\bar{\mathbf{x}}) \\ &= \begin{vmatrix} h_{\mu_{n+1-i}+i-j}(\bar{\mathbf{x}}) \\ \cdots \\ h_{\lambda_{i-n}-i+j}(\mathbf{x}) \end{vmatrix} = \begin{vmatrix} e_{\mu'_{n+1-i}+i-j}(\bar{\mathbf{x}}) \\ \cdots \\ e_{\lambda'_{i-n}-i+j}(\mathbf{x}) \end{vmatrix}. \end{aligned} \tag{73}$$

5.3. Combinatorial proofs

Bressoud and Wei prove their result using a combinatorial argument involving lattice paths. We have extended their techniques to prove our two main results combinatorially by means of lattice path methods. Full details are to be made available in a future paper [3].

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